**Single species population model:** Let  be the population of a species at time . Then the general form of a single species population model is



This is also called conservation equation for the population. The simplest model has no migration and the birth and death terms are proportional to .

**Continuous growth models:** There are some continuous growth models such as:

1. **Malthusian model:** The Malthusian model is



where is proportionality constant and .

1. **Logistic Model:** The Logistic model or Pearl-Verhulst model is



where is carrying capacity and is a positive constant.

1. **Delay Model:** If the birth rate of a population is considered to act instantaneously whereas there may be a time of delay to take account of the time to reach maturity, the finite gestation period and so on, then the delay model is



where is delay parameter.

1. **Harvesting Model:** The harvesting model is



where and  are positive constants,  is the linear growth rate,  is the natural carrying capacity and  is the harvesting.

**Carrying Capacity:** Ifan environment is capable of sustaining no more than a fixed number  of individuals in its population, then  is called the carrying capacity of the environment.

**Question-01:** Describe the Malthusian model for the dynamics of a single species population and comment on its plausibility. Discuss the stability of the equilibrium states of the model.

**Answer:** We consider a population of single species. The growth rate of species is proportional to the number of species at any time . The proportionality constant may be dependent or independent of the number of species and time. Let be the number of population of the species. Then the first order linear ordinary differential equation



where,  is a constant and , is called the Malthusian model.

From (1), we get



Integrating, 



where  is an integrating constant.

If at the initial population is  , then from (2) we get





Thus the solution of (1) is given by



Figure: Malthusian model

This represents the population at any time . It shows that the population grows exponentially if , decays exponentially if  and remains constant if .

**Stability:** If the model (1) is stable, if the model (1) is unstable and if the model (1) is

asymptotically stable.

Since  if 

 if 

 if 

The maximum population is



Here  is the equilibrium or critical point.

**Comment:** From the Malthusian model, we observe that the population increases exponentially with

time and  as . Thus the growth is unlimited. This model is remarkably accurate in the case of human population of the earth during the last several decades. It is also outstanding accurate for comparatively smaller size of the population under short period of time. But it is unrealistic when applied to distant future and completely unrealistic for large size of population used over sufficiently long period of time. For in reality after a certain period of time, the population attains a maximum size and remains constant.

**Question-02:** Describe the Logistic model or Pearl-Verhulst model for the dynamics of a single species population and comment on its plausibility. Discuss the stability of the equilibrium states of the model.

**OR**

* Interpret the parameters  and  in the logistic growth model  for a single species

population. Obtain a formula for the population size . Determine the steady states and discuss their stability.

**OR**

* Suppose that a certain population obeys the Verhulst model with intrinsic growth rate and carrying

capacity . Find the complete solution of the model. Discuss also the limiting behavior of the model as .

**Answer:** We consider a population of single species and let be the number of population of the species at any time . Then the first order non-linear ordinary differential equation





where  , is called the Logistic model or Pearl-Verhulst model. Here  is called carrying capacity,  and  are positive constants and  is very small compared to .

The equation (1) is a separable or Bernoulli type equation. So separating the variables, we get







Integrating, 



where  is an integrating constant.

If at the initial population is  , then from (2) we get



Putting this value in (2), we get

























This represents the population at any time . It is complete solution of the model.

The maximum population is





Figure: Logistic model

From the graph, we observe that if the population increases and approaches to  as . Ifthe population decreases and approaches to  as . The population remains constant when . Thus it is clear that  is a limiting factor or limiting behaviour on the growth of the population.

**Stability:** The equilibrium or critical points or steady states of the logistic model are given by







 and 

For linearization about  we put  with  in (1). Thus we have



Since  so neglecting  term, we get







This shows that  as . Thus is an unstable equilibrium point.

Again for linearization about  we put  with  in (1). Thus we have







Since  so neglecting  term, we get





This shows that  as . Thus the equilibrium  i.e. is stable.

**Comment:** From the Logistic model, we observe that the population does not increase exponentially

with time and  as . Thus  is a limiting factor on the growth of the population. This model is remarkably accurate in the case of human population of the earth during the last several decades.

**Question-03:** Explain a single species population model together with its stability and equilibrium point.

**Answer:** The logistic growth model



is a single species population model, where  and  are positive constants and is intrinsic growth rate, is the carrying capacity .

For stability and equilibrium we have







 and 

Thus there are two equilibrium states

 and 

For linearization about  we put  with  in (1). Thus we have



Since  so neglecting  term, we get







This shows that  as . Thus is an unstable equilibrium point.

Again for linearization about  we put  with  in (1). Thus we have







Since  so neglecting  term, we get





This shows that  as . Thus the equilibrium  i.e. is stable.

**Question-04:** Describe the Harvesting model for the dynamics of a single species population.

**Problem**

**Problem-01:** A population  grows accordingly to the Malthus law , where  is a positive constant. Determine how long it takes the population to double in size.

**Solution:** We have





Integrating, 



where  is an integrating constant.

Suppose the initial population is  at , then from (1) we get





Substituting this value in (1) we get



Let the population will be double i.e.  at time .

From (2), we get







.

This is the required time.

**Problem-02:** Assume that the population of the Cumilla city increases at a rate proportional to the number of inhabitants at any time. If the population doubles in 40 years in how many years will it be triple?

**Solution:** Let  be the number of inhabitants at any time  and  be the initial population at time . Now according to the question, we have





Integrating, 



where  is an integrating constant.

Since  at , then from (1) we get





Substituting this value in (1) we get



Again since  at , then from (2) we get











Let the population will be triple i.e.  at time .

From (2), we get









 years.

This is the required time.

**Problem-03:** Assume that the population of the Cumilla city increases at a rate proportional to the number of inhabitants at any time. If the population was 30,000 in 1970 and 35,000 in 1980. What will be the population in the year 2000 and 1990?

**Solution:** Let  be the number of inhabitants at any time . Now according to the question, we have





Integrating, 



where  is an integrating constant.

Since  at , then from (1) we get



Again since  at , then from (1) we get



Dividing (3) by (2), we get









Putting the value of  in (2), we get







Substituting the value of  and  in (1), we get



Now at the population will be



 (Ans)

Again at the population will be



 (Ans)

**Problem-04:** The world population was estimated to be 1550 millions in 1900 and 2500 millions. Estimate the population at the world in the year 2000 by using Malthusian law.

**Solution:** Let  be the number of population at any time . Now according to the question, we have





Integrating, 



where  is an integrating constant.

Since  at , then from (1) we get



Again since  at , then from (1) we get



Dividing (3) by (2), we get









Putting the value of  in (2), we get







Substituting the value of  and  in (1), we get



Now at the population will be





millions (Ans)

**Problem-05:** Bangladesh population was estimated to be 800 millions in 1980 and 1200 millions in 2000. Estimate the population of Bangladesh in the year 2020 by using Malthusian law.

**Solution:** Let  be the number of population of Bangladesh at any time . Now according to the question, we have





Integrating, 



where  is an integrating constant.

Since  at , then from (1) we get



Again since  at , then from (1) we get



Dividing (3) by (2), we get









Putting the value of  in (2), we get







Substituting the value of  and  in (1), we get



Now at the population will be





millions (Ans)

**Problem-06:** If the population of a country doubles in 50 years in how many years will it triple under the assumption of the Malthusian model? What will be the population in the year 2020?

**Solution:** Let  be the number of population of the country at any time  and  be the initial population at time . Now according to the question, we have





Integrating, 



where  is an integrating constant.

Since  at , then from (1) we get





Substituting this value in (1) we get



Again since  at , then from (2) we get











Let the population will be triple i.e.  at time .

From (2), we get







 years.

This is the required time.

At  the population will be

 (Ans)

**Problem-07:** The population of Cumilla satisfies the logistic law , where the time  is measured in years. If the population of Cumilla was 200000 in 1980, then what will be the population in the year 2000? What would be the possible maximum population of Cumilla?

**Solution:** We have



and 

From (1), we have









Integrating, 











where  is an integrating constant.

Using (2) in (3), we get









Putting this value in (3), we get







At the population will be





Hence the population of Cumilla in the year 2000 will be 312965.

The maximum population is









Hence the maximum population of Cumilla will be 1000000.

**Problem-08:** The population (in million)of the USA satisfies the logistic law ,

, , , where the time  is measured in years. Find the maximum population of USA and the population expected in the year 2020.

**Solution:** We have



and 

From (1), we have







Integrating, 

where  is an integrating constant.

Using  for  in (3), we get



From (3) and (4), we get

















The maximum population is









Hence the maximum population of USA will be .

Let  correspond in the years 1930, 1960, 1990 respectively.

Then from (2) and (5), we get









and 



From (8) and (7), we get



From (9) and (7), we get



Dividing (11) by (10), we get













From (10) and (12), we get









The maximum population is

 millions

Putting the values of ,  and  in (5), we get



This is a formula for all future population.

Now corresponds to the year 2020 and putting in (13), we get



 millions

Thus the expected population in the year 2020 is 279.68 millions.

**Problem-09:** The population of a certain city satisfies the logistic law , where the time  is measured in years. If the population of the city was 100,000 in 1980, then determine the population as a function of time for . What will be the population in 2000? In which year will the population be double? How large will the population be in size?

**Solution:** We have



and 

From (1), we have







Integrating, 











where  is an integrating constant.

Using (2) in (3), we get









Putting this value in (3), we get







This is the required population as a function of time for .

Now at the population will be





Hence the population of the city in the year 2000 will be 168369.88.

Let the population will be double i.e. for .

Then from (4), we get













 years.

The maximum population is









Hence the maximum population of Cumilla will be 1000000.

**Problem-10:** A population grows accordingly to the equation , where  and  are positive constants. Find the stability of the equilibrium population. Determine the equilibrium population size.

**Answer:** We have



where  and  are positive constants.

For equilibrium population we have















Hence the equilibrium population is .

For linearization about  we put  with  in (1). Thus we have







Since  so neglecting higher order terms, we get





This shows that  as . Thus the equilibrium  i.e. is stable.

**Problem-11:** A certain population  obeys the logistic model with specific growth rate  and the carrying capacity . Find the complete solution of the model. Discuss the behavior of the population as . Obtain a formula for the time , when the population size  and the initial population is  with .

**Solution:** We know, the logistic model is



where is the number of population at any time ,  and  are positive constants and  is very small compared to ,  is carrying capacity.

The equation (1) is a separable or Bernoulli type equation. So separating the variables, we get







Integrating, 



where  is an integrating constant.

If at the initial population is  , then from (2) we get



Putting this value in (2), we get

























This represents the population at any time . It is complete solution of the model.

When then we have







Therefore we observe that, there is a limit to the growth of as required by biological fact. The maximum population is .

Again, for the time  the population is given by  and the initial population is .

From (3), we have



















This is the required formula for the time .